MATH 430, SPRING 2022
NOTES MARCH 30 - APRIL 8

Below $\mathfrak{A} = (\mathbb{N}, <, S, +, \cdot, 0, 1)$ is the standard model of PA. By compactness, PA has nonstandard models i.e. a model $\mathcal{B} \models PA$, such that $\mathcal{B} \ncong \mathfrak{A}$. (See homework problem)

Now, suppose that $\mathcal{B}$ is a nonstandard model of PA.

Theorem 1. *$\mathcal{B}$ is an end extension of $\mathfrak{A}$ i.e. there is a one-to-one homomorphism $f : \mathbb{N} \to |\mathcal{B}|$ such that for all $a \in \text{ran}(f)$ and $b \in |\mathcal{B}| \setminus \text{ran}(f)$, $a <^{\mathcal{B}} b$.*

Proof. Define f by $f(n) = (S^{\mathcal{B}})^n(0^{\mathcal{B}})$. f is one-to-one, since $PA \models S$ is one-to-one. Next we can verify that f is a homomorphism:

- $f(0) = 0^{\mathcal{B}}$ and $f(1) = S^{\mathcal{B}}(0^{\mathcal{B}}) = 1^{\mathcal{B}}$, by definition;
- $n < m$ iff $f(n) <^{\mathcal{B}} f(m)$, this is because $PA \models S$ is order preserving.
- $f(S(n)) = S^{\mathcal{B}}(f(n))$, this is because $PA \models S^{n+1}(0) = S(S^n(0))$;
- $f(n+m) = f(n) + f(m)$, this is because $PA \models S^{n+m}(0) = S^n(0) + S^m(0)$;
- $f(n \cdot m) = f(n) \cdot f(m)$, this is because $PA \models S^{n \cdot m}(0) = S^n(0) \cdot S^m(0)$;

Finally, suppose that $a \in \text{ran}(f)$ and $b \in |\mathcal{B}| \setminus \text{ran}(f)$. We have to show that $a <^{\mathcal{B}} b$. Since $a \in \text{ran}(f)$, by definition of f, for some n, $a = f(n)$. We show that $a = f(n) <^{\mathcal{B}} b$ by indiction on n:

(1) if $n = 0$, since $PA \models 0$ is the least element, we have that $f(0) = 0^{\mathcal{B}} < b$.
(2) $n = k + 1$ and inductively we assume that $f(k) <^{\mathcal{B}} b$. Then since $PA \models \forall x \forall y (x < y \to (S(x) < y \vee S(x) = y))$, we have that either $f(k+1) = S^{\mathcal{B}}(f(k)) = b$ of $f(k+1) <^{\mathcal{B}} b$. Since $b \notin \text{ran}(f)$, it cannot equal $f(k+1)$. So $f(k+1) <^{\mathcal{B}} b$.

□

It follows that $\mathcal{B}$ contains an isomorphic copy of the natural numbers as an initial segment. For simplicity of notation, write n to denote $(S^{\mathcal{B}})^n(0^{\mathcal{B}})$. For example, we write 0 for $0^{\mathcal{B}}$, 1 for $S^{\mathcal{B}}(0) = 1^{\mathcal{B}}$, 2 for $S^{\mathcal{B}}(S^{\mathcal{B}}(0))$ and so on.

Similarly, for a formula $\phi(x_1, ..., x_n)$ and $a_1, ..., a_n$ in $\mathbb{N}$, we say that $PA \models \phi[a_1, ..., a_n]$, if for any model $\mathcal{B} \models PA$, $\mathcal{B} \models \phi[a_1, ..., a_n]$.

Definition 2. *Let ϕ be a formula ϕ in the language of PA.*

(1) *ϕ is Δ_0 if it is logically equivalent to a formula with only bounded quantifiers (or no quantifiers).*
(2) *ϕ is Σ_1 if it is logically equivalent to a formula of the form $\exists x_1, ..., \exists x_n \psi$, where ψ is Δ_0.*

(3) *ϕ is Π_1 if it is logically equivalent to a formula of the form $\forall x_1, ..., \forall x_n \psi$, where ψ is Δ_0.*
(4) *ϕ is Δ_1 if it is logically equivalent to both a Σ_1 and a Π_1 formula.*

Examples of Δ_0 formulas:

- all atomic formulas;
- $\phi_{div}(x, y) = \exists z < y(x \cdot z = y)$;
- $\phi_{prime}(x) = x > 1 \wedge \forall z < x(\phi_{div}(z, x) \to z = 1)$.

Note that if ϕ is Σ_1, then $\neg\phi$ is Π_1. Similarly, if ϕ is Π_1, then $\neg\phi$ is Σ_1 Also, if both ϕ and $\neg\phi$ are Σ_1, then ϕ is Δ_1.

Theorem 3. *If $\mathcal{B} \models PA$, and $\phi(x_1, ..., x_n)$ is a Δ_0-formula, then for any $a_1, ..., a_n$ in $\mathbb{N}$, $\mathfrak{A} \models \phi[a_1, ...a_n]$ iff $\mathcal{B} \models \phi[a_1, ...a_n]$.*

Proof. This is by induction on the complexity of ϕ, using that $\mathcal{B}$ is an end extension of $\mathfrak{A}$. Fix $\phi(x_1, ..., x_n)$ and natural numbers $a_1, ..., a_n$.

For the base case, if ϕ is atomic, by the existence of the function f in the proof of theorem 1, it follows that $\mathfrak{A} \models \phi[a_1, ...a_n]$ iff $\mathcal{B} \models \phi[a_1, ...a_n]$.

If ϕ is a negation or a conjunction, the statement follows by the inductive hypothesis.

Now, suppose that ϕ is of the form $\forall y < x_1 \psi(x_1, ..., x_n, y)$, where the inductive hypothesis holds for ψ. Then:
$\mathfrak{A} \models \phi[a_1, ...a_n]$ iff
for all $b < a_1$, $\mathfrak{A} \models \psi[a_1, ...a_n, b]$ iff, by the inductive hypothesis,
for all natural numbers $b < a_1$, $\mathcal{B} \models \psi[a_1, ...a_n, b]$.

For any $c \in |\mathcal{B}|$, if $c < a_1$, then $c \in \text{ran}(f)$ i.e. it is a natural number. So, for all natural numbers $b < a_1$, $\mathcal{B} \models \psi[a_1, ...a_n, b]$ iff
$\mathcal{B} \models \phi[a_1, ...a_n]$. □

As a corollary, one can show (the details are in your homework)

Corollary 4. *Suppose $\mathcal{B} \models PA$ and $a_1, ..., a_n$ are in $\mathbb{N}$. Then*

(1) *If $\phi(x_1, ..., x_n)$ is a Σ_1-formula, if $\mathfrak{A} \models \phi[a_1, ...a_n]$, then $\mathcal{B} \models \phi[a_1, ...a_n]$.*
(2) *If $\phi(x_1, ..., x_n)$ is a Π_1-formula, if $\mathcal{B} \models \phi[a_1, ...a_n]$, then $\mathfrak{A} \models \phi[a_1, ...a_n]$.*
(3) *If $\phi(x_1, ..., x_n)$ is a Δ_1-formula, then $\mathfrak{A} \models \phi[a_1, ...a_n]$ iff $\mathcal{B} \models \phi[a_1, ...a_n]$.*
(4) *If $\phi(x_1, ..., x_n)$ is a Σ_1-formula, then $\mathfrak{A} \models \phi[a_1, ...a_n]$ iff $PA \models \phi[a_1, ...a_n]$.*

Proof. For the first three items, the proofs are assigned as homework. For the last one, fix a Σ_1-formula $\phi(x_1, ..., x_n)$. For the first (easy) direction, suppose that $PA \models \phi[a_1, ...a_n]$. Then since $\mathfrak{A} \models PA$, we have that $\mathfrak{A} \models \phi[a_1, ...a_n]$.

For the other direction, suppose that $\mathfrak{A} \models \phi[a_1, ...a_n]$. Then by item (1) above any for any model $\mathcal{B}$ of PA, we have that $\mathcal{B} \models \phi[a_1, ...a_n]$. It follows that $PA \models \phi[a_1, ...a_n]$.

□

The **primitive recursive functions** are (total) functions $f : \mathbb{N}^k \to \mathbb{N}$, that are built up from the constant function $f(x) = 0$, projections, and the successor function S, by applying composition and the primitive recursion operation:

- $f(0, a_1, ..., a_n) = g(a_1, ..., a_n)$;
- $f(n + 1, a_1, ..., a_n) = h(n, f(n, a_1, ..., a_n), a_1, ..., a_n)$;

where g, h are primitive recursive.

Examples of primitive recursive functions: addition, multiplication, exponentiation.

Later we will show the following theorem.

Theorem 5. *Suppose that $f : \mathbb{N}^k \to \mathbb{N}$ is primitive recursive. Then there is a Δ_1 formula $\phi(x_0, ..., x_{k-1}, y)$, such that for all $a_1, ..., a_{k-1}, b$ in $\mathbb{N}$,*

$$f(a_1, ..., a_{k-1}) = b \text{ iff } \mathfrak{A} \models \phi[a_1, ..., a_{k-1}, b].$$